

Cooperative Games with Minimum Payoffs

Francisco Sánchez Sánchez

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Games in characteristic function form



Teorema (Shapley, 1953)

There exists a unique solution $Sh : G^N \rightarrow \mathbb{R}^n$ that satisfies additivity, symmetry, efficiency and nullity axioms. Furthermore it is given by

$$Sh_i(v) = \sum_{\substack{S \subset N \\ S \not\ni i}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$



Definición

For a given $\alpha \in (0, \frac{1}{n}]$, the solution φ is said to satisfy α -minimum payoff in $H \subseteq G^N$ if and only if

$$\varphi_i(v) \geq \alpha v(N)$$

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Let G^N be the set of all cooperative games. Then for all $\alpha \in (0, \frac{1}{n}]$, there exists a unique linear solution satisfying α -minimum payoff in G^N . Furthermore, it is the egalitarian solution

$$\varphi_i(v) = \frac{v(N)}{n}$$

for all $i \in N$ and $v \in H$.

Games in characteristic function form

- ▶ Let

$$G_+^N = \{v \in G^N \mid v \text{ superadditive, } v(\{i\}) \geq 0 \forall i \in N\}$$

be the cone of non-negative superadditive games.

- ▶ If $v \in G_+^N$ then $Sh(v) \geq 0$.
- ▶ Axiom [Quasi-nullity] The solution φ is said to be quasi-null *with respect to* $\alpha \in (0, \frac{1}{n}]$ if and only if

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for every null player i in $v \in G^N$.

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Teorema

For all $\alpha \in (0, \frac{1}{n}]$, there exists a unique solution satisfying additivity, symmetry, efficiency and quasi-nullity axioms, and it is the value φ defined for all $v \in G^N$ and all $i \in N$, by

$$\varphi_i(v) = (1 - \alpha n)Sh_i(v) + \alpha v(N)$$



Corollary

The solution defined for all $i \in N$ by

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satisfies α -minimum payoff in G_+^N .



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- Moving α from $\frac{1}{n}$ to 0 we get a continuous transition from our solution approaching Shapley's value.

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Games with coalition structures



Definición

A coalition structure \mathfrak{B} is a finite partition $\mathfrak{B} = \{B_1, \dots, B_m\}$ of the set of players (i.e., $\bigcup_{k=1}^m B_k = N$, $S_k \neq \emptyset \forall k$ and $B_k \cap B_j = \emptyset \forall k \neq j$). The set of all coalition structures of N is denoted by PT . A game with coalition structure \mathfrak{B} is a pair $(\mathfrak{B}, v) \in PT \times G^N$.

- ▶ Given N and \mathfrak{B} , Aumann and Drèze (1974) proposed and characterized a “ \mathfrak{B} -value”, which is a function $\phi : PT \times G^N \rightarrow \mathbb{R}^n$ obeying the following conditions.

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- ▶ Axiom [Relative efficiency] For all k , $[\phi(\mathfrak{B}, v)](B_k) = v(B_k)$.
- ▶ Axiom [Symmetry] For all permutations θ of N under which \mathfrak{B} is invariant,

$$[\phi(\mathfrak{B}, \theta \cdot v)](S) = [\phi(\mathfrak{B}, v)](\theta(S))$$

where $(\theta \cdot v)(S) = v(\theta(S))$.

- ▶ Axiom [Additivity] $\phi(\mathfrak{B}, v + w) = \phi(\mathfrak{B}, v) + \phi(\mathfrak{B}, w)$ for all $(\mathfrak{B}, v), (\mathfrak{B}, w) \in PT \times G^N$.
- ▶ Axiom [Nullity] Let $(\mathfrak{B}, v) \in PT \times G^N$ and $i \in N$. If i is a null player in v , then $\phi_i(\mathfrak{B}, v) = 0$.

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Teorema (Aumann and Drèze, 1974)

Fix N and $\mathfrak{B} = \{B_1, B_2, \dots, B_m\}$. Then there is a unique value ϕ satisfying relative efficiency, symmetry, additivity and nullity axioms, and it is given for all $k = 1, 2, \dots, m$, and all $i \in B_k$, by

$$\phi_i(\mathfrak{B}, v) = Sh_i(v|_{B_k})$$

Games with coalition structures



Definición

Let $(\mathfrak{B}, v) \in PT \times G^N$, where $\mathfrak{B} = \{B_1, B_2, \dots, B_m\}$, and a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ with $\alpha_k \in \left(0, \frac{1}{b_k}\right]$. We will say that a solution φ satisfies α -minimum payoff in $H \subseteq G^N$ if and only if

$$\varphi_i(\mathfrak{B}, v) \geq \alpha_k v(B_k)$$

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There exists a unique solution $\varphi : PT \times G^N \rightarrow \mathbb{R}^n$ satisfying relative efficiency, symmetry, additivity and quasi-nullity axioms, and is given for all $k = 1, 2, \dots, m$, and all $i \in B_k$, by

$$\varphi_i(\mathfrak{B}, v) = (1 - \alpha_k b_k) \phi_i(\mathfrak{B}, v) + \alpha_k v(B_k)$$



Corollary

The solution

$$\varphi_i(\mathfrak{B}, v) = (1 - \alpha_k b_k) \phi_i(\mathfrak{B}, v) + \alpha_k v(B_k)$$

satisfies α -minimum payoff in G^{B_k} for all $k = 1, 2, \dots, m$ and all $i \in B_k$.

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Games in partition function form

- ▶ Given N , let

$$ECL = \{(S, Q) \mid S \in Q \in PT\}$$

be the set of *embedded coalitions*.



Definición

A mapping

$$w : ECL \rightarrow \mathbb{R}$$

that assigns a real value, $w(S, Q)$, to each embedded coalition (S, Q) is called a game in partition function form. The set of games in partition function form with player set N is denoted by PG , i.e.,

$$PG = \{w \mid w : ECL \rightarrow \mathbb{R}\}$$

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- ▶ The value $w(S, Q)$ represents the payoff of coalition S , given the coalition structure Q forms. For any $w \in PG$ and any $Q \in PT$, we adopt the convention that $w(\emptyset, Q) = 0$.

- ▶ For any $Q, \bar{Q} \in PT$, define

$$Q \wedge \bar{Q} = \{S \cap \bar{S} \mid S \in Q, \bar{S} \in \bar{Q}, S \cap \bar{S} \neq \emptyset\}$$

- ▶ Axiom [Additivity] The solution Φ is additive if

$$\Phi(w_1 + w_2) = \Phi(w_1) + \Phi(w_2)$$

for all $w_1, w_2 \in PG$.

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- ▶ Axiom [Symmetry] The solution Φ is said to be symmetric if and only if $\Phi_i(w) = \Phi_{\theta(i)}(\theta \cdot w)$ for every $\theta \in S_n$, every $w \in PG$ and every $i \in N$, where the game $\theta \cdot w$ is defined as

$$(\theta \cdot w)(S, Q) = w[\theta^{-1}(S, Q)]$$

- ▶ Given $w \in PG$ and $\bar{S} \in 2^N \setminus \{\emptyset\}$, we say that \bar{S} is a *carrier* of w if and only if $w(S, Q) = w(S \cap \bar{S}, Q \wedge \{\bar{S}, N \setminus \bar{S}\})$. The next axiom suggests that all available wealth should be divided among the members of a carrier.
- ▶ Axiom [Carrier] For all $w \in PG$ and all $S \in 2^N \setminus \{\emptyset\}$, if S is a carrier of w , then

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- ▶ Axiom [Carrier] For all $w \in PG$ and all $S \in 2^N \setminus \{\emptyset\}$, if S is a carrier of w , then

$$\sum_{i \in S} \Phi_i(w) = w(N, \{N\})$$

Games in partition function form

Teorema (Myerson, 1977)

The solution $\Phi : PG \rightarrow \mathbb{R}^n$ given by

$$\Phi_i(w) = \sum_{(S,Q) \in ECL} (-1)^{q-1} (q-1)! \cdot \left(\frac{1}{n} - \sum_{T \in Q \setminus \{S\}, i \notin T} \frac{1}{(q-1)(n-t)} \right) \cdot w(S, Q)$$

for each $i \in N$ and each $w \in PG$; is the unique solution satisfying additivity, symmetry and carrier axioms.

Games in partition function form



Definición

For a given $\alpha \in (0, \frac{1}{n}]$, the solution φ is said to satisfy α -minimum payoff in $H \subseteq PG$ if and only if

$$\varphi_i(w) \geq \alpha w(N, \{N\})$$

for all $i \in N$ and $w \in H$.

- ▶ Axiom [Quasi-carrier] For all $w \in PG$ and all $S \in 2^N \setminus \{\emptyset\}$, if S is a carrier of w , then

$$\sum_{i \in S} \Phi_i(w) = (1 - \alpha(n - s))w(N, \{N\})$$

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Games in partition function form



Teorema

For all $\alpha \in (0, \frac{1}{n}]$, there exists a unique solution satisfying additivity, symmetry and quasi-carrier axioms, and it is the value φ defined for all $w \in PG$ and all $i \in N$, by

$$\varphi_i(w) = (1 - \alpha n)\Phi_i(w) + \alpha w(N, \{N\})$$



Corollary

The solution defined for all $i \in N$ by

$$\varphi_i(w) = (1 - \alpha n)\Phi_i(w) + \alpha w(N, \{N\})$$

satisfies α -minimum payoff in $H = \{w \in PG \mid \Phi(w) \geq 0\}$.

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Games with cooperation graphs

Lets denote by,

- ▶ $g^N = \{\{i, j\} \mid i, j \in N, i \neq j\}$
- ▶ $GR = \{g \mid g \subseteq g^N\}$.
- ▶ Given $g \in GR$ and $S \subseteq N$,

$$S/g = \{\{i \mid i \text{ and } j \text{ are connected in } S \text{ by } g\} \mid j \in S\}$$

Games with cooperation graphs



Definición

An allocation rule for $v \in G^N$ is any function $Y : GR \rightarrow \mathbb{R}^n$ such that

$$\sum_{i \in S} Y_i(g) = v(S)$$

for every $g \in GR$ and every $S \in N/g$.



Definición

An allocation rule is fair if and only if

$$Y_i(g) - Y_i(g \setminus \{i, j\}) = Y_j(g) - Y_j(g \setminus \{i, j\})$$

for every $g \in GR$ and every $\{i, j\} \in g$.

Games with cooperation graphs

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Games with cooperation graphs

Teorema (Myerson, 1977)

Given a characteristic function game $v \in G^N$, there is a unique fair allocation rule $Y : GR \rightarrow \mathbb{R}^n$. This fair allocation rule satisfies, for every $g \in GR$,

$$Y(g) = Sh(v/g)$$

where the game $(v/g) \in G^N$ is defined for all $S \subseteq N$, as

$$(v/g)(S) = \sum_{T \in S/g} v(T).$$

Games with cooperation graphs



Definición

Given $v \in G^N$ and $\alpha \in (0, \frac{1}{n}]$, we say that a rule $Z : GR \rightarrow \mathbb{R}^n$ satisfies minimum payoff in $H \subseteq GR$ if and only if

$$Z_i(g) \geq \alpha(v/g)(N)$$

for every $g \in GR$ and $i \in N$.



Definición

A semi-allocation rule with respect to $\alpha \in (0, \frac{1}{n}]$ for $v \in G^N$, is a function $Z : GR \rightarrow \mathbb{R}^n$ such that

$$\sum_{i \in S} Z_i(g) = v(S) + \alpha [s(v/g)(N) - nv(S)]$$

Games with cooperation graphs

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Games with cooperation graphs



Teorema

Given $v \in G^N$ and $\alpha \in (0, \frac{1}{n}]$, there is a unique fair semi-allocation rule with respect to α , $Z : GR \rightarrow \mathbb{R}^n$. It is given by

$$Z_i(g) = (1 - \alpha n)Sh_i(v/g) + \alpha(v/g)(N)$$

for every $i \in N$ and every $g \in GR$.



Corollary

Given $v \in G^N$ and $\alpha \in (0, \frac{1}{n}]$, the semi-allocation rule with respect to α

$$Z_i(g) = (1 - \alpha n)Sh_i(v/g) + \alpha(v/g)(N)$$

satisfies α -minimum payoff in $H = \{g \in GR \mid v/g \in G_+^N\}$.

Moreover, if $v \in G_+^N$, then the semi-allocation rule with respect to α , Z , 

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